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Thermal receptivity of unstable laminar flow with heat transfer

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Abstract—In this study, open questions about the temperature disturbances in linear stability theory are addressed. Temperature effects on flow stability have been investigated previously either by assuming a time independent shape for the temperature disturbance or by neglecting temperature fluctuations completely. Treating temperature fluctuations as an initial value problem can show the limits in which these assumptions are justified. By means of an asymptotic approach, based on small heat transfer rates, the various effects can be well separated from each other. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Among the numerous studies on flow stability, only a few address the problem of additional heat transfer and its effects on the stability of the flow. In an early study, Wazzan *et al.* [1] found that the critical Reynolds number of a flat plate boundary layer in water under the effect of heat transfer varies between 520 and 16 000 (based on the displacement thickness). Thus, there is a considerable potential for transition control with applications in various fields: see for example Morkovin and Reshotko [2] for an overview.

Since the transition process can best be controlled in its early stage of development, a theoretical approach based on the linear stability analysis is sufficient and adequate.

Physically heat transfer affects the flow stability through the temperature dependence of the properties ρ^* (density), μ^* (viscosity), k^* (thermal conductivity) and c_p^* (specific heat). Basically, there are two ways this temperature dependence can be taken into account.

(1) By inserting the specific temperature dependence of ρ^*, μ^*, k^* and c_p^* of a particular fluid into the stability equations for variable properties. This way of treating the problem is followed by most authors. Often certain dependencies are neglected, assuming for example only μ^* to be a function of temperature. This method from now on will be called 'the direct solution method'. Its shortcoming is that results only hold for the particular fluid under consideration. Typical studies of this kind are those of Asfer *et al.* [3] and Lee *et al.* [4].

(2) By using Taylor series expansions for all the properties with respect to temperature. In this asymptotic approach all effects are well separated from each other and only the Reynolds and Prandtl numbers remain as parameters. In their general form, the asymptotic solutions hold for all Newtonian fluids. This method will be called 'the property expansion method'. It was introduced by our group some years ago in Herwig and Schäfer [5]. Further studies based on this method are Schäfer and Herwig [6] and Schäfer *et al.* [7].

Assuming all variables including temperature can be decomposed into a mean and a superimposed fluctuating part, then heat transfer affects the flow stability in two ways. Firstly, through the influence of the mean temperature $\bar{T}^*$ and also through that of temperature fluctuation $\tilde{T}^*$.

The influence of the temperature fluctuations $\tilde{T}^*$ gave rise to a controversial discussion in the past. Some studies, such as Wazzan *et al.* [1] for flat plate flow or Potter and Graber [8] for plane Poiseuille flow, simply neglected them. As described in Strazisar *et al.* [9], Lowell [10] reformulated Wazzan's problem and included temperature fluctuations. The only justification for neglecting the $\tilde{T}^*$ effects is the fact that at least in some cases they are small. However, it cannot be justified in general. For plane Poiseuille flow, for example, it can be shown that this effect vanishes in the limit of infinite Reynolds numbers, but not for finite values of *Re*, see Schäfer and Herwig [6].

Regardless of the conclusions about the influence of $\tilde{T}^*$, they are all based upon a fundamental assumption about $\tilde{T}^*$: it is assumed that the temperature disturbance $\tilde{T}^*$ in the framework of the linear stability theory has exactly the same form as the velocity disturbance, namely:

$$\tilde{a}^*(x^*,y^*,t^*) = \hat{a}^*(y^*)\exp[i\alpha^*(x^*-\hat{c}^*t^*)]+\text{c.c.} \qquad (1)$$

Details will be described later. At this stage of the analysis it is only important that $\tilde{u}^*$, $\tilde{v}^*$ and $\tilde{T}^*$ are assumed to be of this general form.

The idea behind this assumption about $\tilde{T}^*$ is the passive character of the temperature fluctuation field. It develops as a consequence of velocity fluctuations

NOMENCLATURE

a	general quantity
$\hat{a}$	shape function of a
b	amplitude function, equation (12)
c_i	amplification rate
c_r	phase velocity
c_p	specific heat at constant pressure
H	half channel height
k	thermal conductivity
K_μ	nondimensional viscosity gradient, equation (6)
p	pressure
Pr	Prandtl number $\mu_R^* c_P^*/k^*$
q_w	wall heat flux
Re	Reynolds number $\rho^* U_R^* H^*/\mu_R^*$
t	time
T	temperature
$\hat{T}$	temperature shape function
u	streamwise velocity
U_R	reference velocity
v	velocity normal to the wall
x, y	Cartesian coordinates.

Greek symbols

α	wavelength parameter
ε	perturbation parameter, equation (6)
μ	viscosity
ψ	stream function
ρ	density.

Superscripts

*	dimensional quantity
$\bar{}$	mean value
~	disturbance quantity
′	derivative with respect to y
^	complex quantity.

Subscripts

i	imaginary part
r	real part
R	reference state
w	wall
0	zero order
1	first order
μ	viscosity effect.

in a field with temperature gradients. If $\hat{a}^*$ is a function of y^* alone, as assumed in equation (1), then we call this the 'shape assumption'. On the other hand $\tilde{T}^*$ is governed by an equation with diffusion terms as will be shown later, i.e. it will take time (or a certain distance downstream) for the temperature fluctuation with an amplitude function $\hat{T}^*(y^*, t^*)$ to reach its final form $\hat{T}^*(y^*)$ according to this shape assumption.

It is only possible to find out if this time (or distance) is short enough so that the form (1) prevails before it is overcome by nonlinear effects later on (or further downstream) by analyzing how the temperature disturbance develops from an initial $\tilde{T}^* = 0$ to its final form. This is the objective of this study.

Since this analysis will tell how temperature fluctuations are adopted by the flow we call this process 'thermal receptivity' in analogy to the process of receptivity of the velocity fluctuations in a flow field.

The analysis will be performed for a plane Poiseuille flow with temperature dependent viscosity. As thermal boundary condition we assume heat flux $q_w^* = \text{const.}$ for $x^* > 0$ which experimentally could be realized by heating the Poiseuille flow with electric resistance wires mounted on the channel walls. Downstream of an adjustment zone the flow will reach a fully developed state. In Fig. 1 typical velocity and temperature profiles are shown in this region. Our stability considerations will refer to this part of the flow field. The analysis is restricted to the case of variable viscosity which, in the framework of the property expansion method, is a rational assumption about the fluid behavior (in the sense that it can be extended

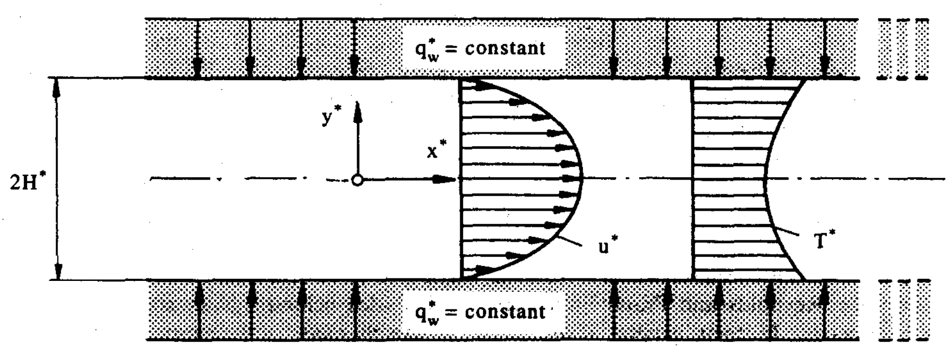

Fig. 1. Velocity and temperature profiles in the fully developed region of a uniformly heated channel.

by additional effects without change of previous results).

2. BASIC EQUATIONS

The present analysis presents an extended version of the linear stability theory which holds when viscosity is a function of temperature. All dimensional quantities are starred and all complex quantities are marked by symbol ^. In the method of small disturbances, which we adopt here, all quantities are decomposed into a mean value, $\bar{a}^*$, and a superimposed disturbance $\tilde{a}^*$. Here, a^* represents the velocity components $u^* = \partial\psi^*/\partial y^*$ and $v^* = -\partial\psi/\partial x^*$ (two-dimensional flow), the pressure p^*, temperature T^* and viscosity μ^*.

From the Navier–Stokes equations and the thermal energy equation (both for variable viscosity), together with the continuity equation, the following linearized differential equations for $\tilde{\psi}^*$ and $\tilde{T}^*$ in their dimensionless form are derived by subtracting the mean flow equations and eliminating the pressure in the momentum equations:

$$\frac{\partial\Delta\tilde{\psi}}{\partial t}+\bar{u}\frac{\partial\Delta\tilde{\psi}}{\partial x}-\frac{\partial^2\bar{u}}{\partial y^2}\frac{\partial\tilde{\psi}}{\partial x}$$
$$=\frac{1}{Re}\left[\bar{\mu}\Delta^2\tilde{\psi}+\tilde{\mu}\frac{\partial^3\bar{u}}{\partial y^3}+4\frac{\partial^2\bar{\mu}}{\partial x\,\partial y}\frac{\partial^2\tilde{\psi}}{\partial x\,\partial y}\right]$$
$$+\frac{1}{Re}\left[2\frac{\partial\bar{\mu}}{\partial x}\frac{\partial\Delta\tilde{\psi}}{\partial x}+2\frac{\partial\bar{\mu}}{\partial y}\frac{\partial\Delta\tilde{\psi}}{\partial y}\right.$$
$$\left.+2\frac{\partial^2\bar{u}}{\partial y^2}\frac{\partial\tilde{\mu}}{\partial y}-\frac{\partial\bar{u}}{\partial y}\left(\frac{\partial^2\tilde{\mu}}{\partial x^2}-\frac{\partial^2\tilde{\mu}}{\partial y^2}\right)\right]$$
$$+\frac{1}{Re}\left[\left(\frac{\partial^2\bar{\mu}}{\partial x^2}-\frac{\partial^2\bar{\mu}}{\partial y^2}\right)\left(\frac{\partial^2\tilde{\psi}}{\partial x^2}-\frac{\partial^2\tilde{\psi}}{\partial y^2}\right)\right] \quad (2)$$

$$\frac{\partial\tilde{T}}{\partial t}+\bar{u}\frac{\partial\tilde{T}}{\partial x}-\frac{1}{RePr}\Delta\tilde{T}=\frac{\partial\bar{T}}{\partial y}\frac{\partial\tilde{\psi}}{\partial x}-\frac{\partial\bar{T}}{\partial x}\frac{\partial\tilde{\psi}}{\partial y} \quad (3)$$

with

$$Re=\frac{\rho^*U_R^*H^*}{\mu_R^*},\quad Pr=\frac{\mu_R^*c_P^*}{k^*},\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

which are, respectively, the Reynolds number, Prandtl number and the Laplace operator. Here, $\bar{u}$ and $\bar{\mu}$ are the temperature dependent mean flow velocity and viscosity, respectively.

The associated boundary conditions for the disturbance are:

$$y=\pm 1:\quad \tilde{\psi}=\frac{\partial\tilde{\psi}}{\partial y}=\frac{\partial\tilde{T}}{\partial y}=0. \quad (4)$$

All equations are nondimensionalized with respect to a reference state R which may be chosen at any position x_R^* in the fully developed region. The reference temperature T_R^* is $\bar{T}^*(x_R^*)$, for details see Schäfer and Herwig [6]. The reference velocity U_R^* is the maximum mean flow velocity with constant viscosity. The nondimensional temperature of the mean flow is $\bar{T}=(\bar{T}^*-T_R^*)/\Delta T_R^*$ and that of the fluctuations is $\tilde{T}=\tilde{T}^*/\Delta T_R^*$ with $\Delta T_R^*=q_w^*H^*/k^*$.

3. PROPERTY EXPANSION METHOD

The basic idea behind the property expansion method is to combine the Taylor series expansion of μ^* (or all properties in the general case) with respect to temperature with a regular perturbation procedure of the whole problem. From the Taylor series expansion of μ^*, which in nondimensional form reads:

$$\mu=\frac{\mu^*}{\mu_R^*}=1+\varepsilon K_\mu T+O(\varepsilon^2) \quad (5)$$

with

$$K_\mu=\left(\frac{\mathrm{d}\mu^*}{\mathrm{d}T^*}\frac{T^*}{\mu^*}\right)_R,\quad \varepsilon=\frac{q_w^*H^*}{k^*T_R^*} \quad (6)$$

a small quantity ε can be extracted which may serve as a perturbation parameter of the whole problem. Truncating the Taylor series after the linear term results in a linear perturbation theory with respect to ε. Extension to higher order ($\varepsilon^2, \varepsilon^3\ldots$) is straightfoward, but not in the scope of this study (for details, see Herwig and Schäfer [5]). The parameter K_μ is a property of the fluid (for example: water at $T_R^*=293$K $K_\mu=-7.13$).

According to equation (5), we expand:

$$\{\bar{u},\bar{T},\bar{\mu}\}^{\mathrm{T}}=\{\bar{u}_0,\bar{T}_0,1\}^{\mathrm{T}}+\varepsilon K_\mu\{\bar{u}_\mu,\bar{T}_\mu,\bar{T}_0\}^{\mathrm{T}}+O(\varepsilon^2) \quad (7)$$

$$\{\tilde{\psi},\tilde{T},\tilde{\mu}\}^{\mathrm{T}}=\{\tilde{\psi}_0,\tilde{T}_0,0\}^{\mathrm{T}}+\varepsilon K_\mu\{\tilde{\psi}_\mu,\tilde{T}_\mu,\tilde{T}_0\}^{\mathrm{T}}+O(\varepsilon^2). \quad (8)$$

In these expansions, fluctuations with index 0 describe the constant property behavior and those with index μ reflect the influence of viscosity deviations due to its temperature dependence.

The mean flow part of the solution, equation (7), i.e. $\bar{u}_0$, $\bar{T}_0$, $\bar{u}_\mu$, and $\bar{T}_\mu$, can be given analytically (see Schäfer and Herwig [6], note that we used a different nondimensionalization for the velocity):

$$\bar{u}_0=(1-y^2)$$
$$\bar{T}_0=\frac{3}{2}\left(-\frac{1}{12}y^4+\frac{1}{2}y^2-\frac{13}{140}\right)+\frac{3x}{2RePr}$$
$$\bar{u}_\mu=-\frac{1}{24}y^6+\frac{3}{8}y^4-\frac{111}{280}y^2+\frac{53}{840}$$
$$\bar{T}_\mu=\frac{1}{2}\left(-\frac{1}{448}y^8+\frac{3}{80}y^6-\frac{111}{1120}y^4\right.$$
$$\left.+\frac{53}{560}y^2+\frac{16\,917}{2\,587\,200}\right). \quad (9)$$

The reference point R is taken at $x = 0$ in our calculations. The disturbance formulation will be treated in the next section.

4. LINEAR STABILITY THEORY: AN INITIAL VALUE PROBLEM FOR TEMPERATURE DISTURBANCES

Inserting the expansions (7) and (8) into the linear stability equations (2) and (3) and collecting terms of equal magnitude gives the equations for $\tilde{\psi}_0$, $\tilde{T}_0$, $\tilde{\psi}_\mu$ and $\tilde{T}_\mu$. However, before we do this, two steps are appropriate.

(1) We split the first-order function $\tilde{\psi}_\mu$, which reflects the change of the velocity disturbance due to temperature, into two parts, i.e. we write:

$$\tilde{\psi}_\mu(T) = \tilde{\psi}_{\mu 1}(\bar{T}) + \tilde{\psi}_{\mu 2}(\tilde{T}). \quad (10)$$

Thus the influence of the temperature disturbance $\tilde{T}$ is well separated from that of the mean temperature $\bar{T}$. This separation is possibly due to the linearity of equation (2).

(2) We make the common assumption about a single oscillation of the Fourier decomposition of an arbitrary two-dimensional disturbance (temporal stability):

$$\tilde{\psi}_0 = b_0(t)\hat{\psi}_0(y)\exp[i(\alpha x - \theta)] + \text{c.c.} \quad (11)$$

This basically is equation (1) of the introductory section. We only extracted the imaginary part c_i of the complex quantity $\hat{c}$, cast it into an amplitude function $b_0(t)$ and introduced $\theta = \alpha c_r t$ with:

$$\frac{db_0}{dt} = \alpha c_{0i} b_0 + \varepsilon K_\mu (A_{11} + A_{12}) \quad (12)$$

$$\frac{d\theta}{dt} = \alpha c_{0r} + \varepsilon K_\mu (B_{11} + B_{12}). \quad (13)$$

In equations (12) and (13), A_{11}, B_{11} and A_{12}, B_{12} are mean temperature and temperature fluctuation effects, respectively. Since there is an arbitrary constant in $\tilde{\psi}_0$, we normalized $\hat{\psi}_0$ according to the condition that $\max|\partial\hat{\psi}_0/\partial y| = 1$.

However, it is important that we assume the functional form (11) only for $\tilde{\psi}_0$, but not for $\tilde{T}_0$, $\tilde{T}_\mu$, $\tilde{\psi}_{\mu 1}$ and $\tilde{\psi}_{\mu 2}$. They may eventually be of the same form, but when and how $\tilde{T}_0$, for example, acquires this form is the objective of this study. Thus, without loss of generality we seek solutions of the form:

$$\{\tilde{T}_0, \tilde{T}_\mu, \tilde{\psi}_{\mu 1}, \tilde{\psi}_{\mu 2}\}^T = \{\hat{T}_0(y,t), \hat{T}_\mu(y,t), \hat{\psi}_{\mu 1}(y,t), \hat{\psi}_{\mu 2}(y,t)\}^T \times \exp[i(\alpha x - \theta)] + \text{c.c.} \quad (14)$$

If now we insert equations (7) and (8) together with equations (12) and (13) into equations (2) and (3), taking into account the assumptions (11) and (14), we end up with the following zero and first order equations with respect to ε:

Zero order

$$i\alpha(\bar{u}_0 - \hat{c}_0)(\hat{\psi}_0'' - \alpha^2\hat{\psi}_0) - i\alpha\bar{u}_0''\hat{\psi}_0 - \frac{1}{Re}(\hat{\psi}_0^{(4)} - 2\alpha^2\hat{\psi}_0'' + \alpha^4\hat{\psi}_0) = 0 \quad (15)$$

$$\left[\frac{\partial}{\partial t} + i\alpha(\bar{u}_0 - c_{0r})\right]\hat{T}_0 - \frac{1}{RePr}(\hat{T}_0'' - \alpha^2\hat{T}_0) = i\alpha b_0\bar{T}_0'\hat{\psi}_0 - \frac{3}{2RePr}b_0\hat{\psi}_0'. \quad (16)$$

First order

$$\left[\frac{\partial}{\partial t} + i\alpha(\bar{u}_0 - c_{0r})\right](\hat{\psi}_{\mu 1}'' - \alpha^2\hat{\psi}_{\mu 1}) - i\alpha\bar{u}_0''\hat{\psi}_{\mu 1} - \frac{1}{Re}(\hat{\psi}_{\mu 1}^{(4)} - 2\alpha^2\hat{\psi}_{\mu 1}'' + \alpha^4\hat{\psi}_{\mu 1})$$
$$= -i\alpha\bar{u}_\mu(\hat{\psi}_0'' - \alpha^2\hat{\psi}_0)b_0 + i\alpha\bar{u}_\mu''\hat{\psi}_0 b_0 + \frac{1}{Re}[\bar{T}_0(\hat{\psi}_0^{(4)} - 2\alpha^2\hat{\psi}_0'' + \alpha^4\hat{\psi}_0)]b_0$$
$$+ \frac{1}{Re}[2\bar{T}_0'(\hat{\psi}_0''' - \alpha^2\hat{\psi}_0') + \bar{T}_0''(\hat{\psi}_0'' + \alpha^2\hat{\psi}_0)]b_0$$
$$+ \frac{3i\alpha}{Re^2 Pr}(\hat{\psi}_0'' - \alpha^2\hat{\psi}_0)b_0 - (A_{11} - ib_0B_{11})(\hat{\psi}_0'' - \alpha^2\hat{\psi}_0) \quad (17)$$

$$\left[\frac{\partial}{\partial t} + i\alpha(\bar{u}_0 - c_{0r})\right](\hat{\psi}_{\mu 2}'' - \alpha^2\hat{\psi}_{\mu 2}) - i\alpha\bar{u}_0''\hat{\psi}_{\mu 2} - \frac{1}{Re}(\hat{\psi}_{\mu 2}^{(4)} - 2\alpha^2\hat{\psi}_{\mu 2}'' + \alpha^4\hat{\psi}_{\mu 2})$$
$$= \frac{1}{Re}\left[\bar{u}_0'''\hat{T}_0 + 2\bar{u}_0''\frac{\partial\hat{T}_0}{\partial y} + \bar{u}_0'\left(\frac{\partial^2\hat{T}_0}{\partial y^2} + \alpha^2\hat{T}_0\right)\right] - (A_{12} - ib_0B_{12})(\hat{\psi}_0'' - \alpha^2\hat{\psi}_0) \quad (18)$$

with the associated boundary conditions:

$$y = \pm 1: \quad \hat{\psi}_0 = \hat{\psi}_0' = \hat{\psi}_{\mu 1} = \hat{\psi}_{\mu 1}' = \hat{\psi}_{\mu 2} = \hat{\psi}_{\mu 2}' = \hat{T}_0 = 0. \quad (19)$$

Here a' denotes the derivative of a with respect to y. The initial conditions for $\hat{T}_0(y,t)$, $\hat{\psi}_{\mu 1}(y,t)$ and $\hat{\psi}_{\mu 2}(y,t)$ are:

$$t = 0: \quad \hat{T}_0 = \hat{\psi}_{\mu 1} = \hat{\psi}_{\mu 2} = 0. \quad (20)$$

Equation (15) is the well-known Orr–Sommerfeld equation in standard form, equations (17) and (18) are its first-order extension to account for variable viscosity by the perturbation solution (8) and (10).

Equations (16) and (19) together with the initial condition (20) are an initial value problem for the temperature disturbance $\tilde{T}(y,t) = \tilde{T}_0(y,t) + O(\varepsilon)$. Note that there is no need for the first-order temperature disturbance $\tilde{T}'_\mu$ due to the asymptotic level of the problem (first-order in ε).

If we would apply the shape assumption also to $\tilde{T}_0$, $\tilde{\psi}_{\mu 1}$ and $\tilde{\psi}_{\mu 2}$ (and not only to $\tilde{\psi}_0$, see equation (11)) instead of equations (16)–(18), we would get equations used in other studies already, see for example Schäfer and Herwig [6]. Compared to equations (16)–(18), in these equations $\hat{T}_0(y,t)$, $\hat{\psi}_{\mu 1}(y,t)$ and $\hat{\psi}_{\mu 2}(y,t)$ would be replaced by $b_0\hat{T}_0(y)$, $b_0\hat{\psi}_{\mu 1}(y)$ and $b_0\hat{\psi}_{\mu 2}(y)$ and $\mathrm{d}b_0/\mathrm{d}t$ would be equal to $\alpha c_{0i} b_0$.

5. SOLUTION PROCEDURE FOR THE INITIAL VALUE PROBLEM

It is well known that in plane Poiseuille flow the most unstable mode $\hat{\psi}_0$, like the mean flow velocity $\bar{u}_0$, is symmetrical. Since, from the right-hand side of equation (16), we know that $\hat{T}_0(y,t)$ is unsymmetrical, we assume:

$$\hat{T}_0(y,t) = \sum_{n=1}^{M+1} (a_{rn}(t) + i a_{in}(t)) T_{2n-1}(y). \quad (21)$$

Here $T_{2n-1}(y) = \cos[(2n-1)\cos^{-1} y]$ are Chebyshev polynomials and $M+1$ is the number of polynomials we use. Inserting equation (21) into equation (16), using the so-called Chebyshev tau method (for details of this method, see Orszag [11]), we get:

$$\left[\frac{\mathrm{d}a_{rn}}{\mathrm{d}t}\right]_{M\times 1} = [c_{rmn}]_{M\times M}[a_{rn}]_{M\times 1} - [c_{imn}]_{M\times M}[a_{in}]_{M\times 1} + b_0[d_{rn}]_{M\times 1} \quad (22)$$

$$\left[\frac{\mathrm{d}a_{in}}{\mathrm{d}t}\right]_{M\times 1} = [c_{rmn}]_{M\times M}[a_{in}]_{M+1} + [c_{imn}]_{M\times M}[a_{rn}]_{M\times 1} + b_0[d_{in}]_{M\times 1}. \quad (23)$$

Here, $[c_{rmn} + i c_{imn}]_{M\times M}$ is the coefficient matrix, $[d_{rn} + i d_{in}]_{M\times 1}$ is the matrix corresponding to the right-hand side of equation (16). Taking into account the boundary conditions (19), a_{rM+1} and a_{iM+1} can be determined as:

$$a_{rM+1} = -\frac{1}{(2M+1)^2}\sum_{n=1}^{M}(2n-1)^2 a_{rn} \quad (24)$$

$$a_{iM+1} = -\frac{1}{(2M+1)^2}\sum_{n=1}^{M}(2n-1)^2 a_{in}. \quad (25)$$

The initial conditions are:

$$t=0: \quad a_{rn} = a_{in} = 0, \quad (n = 1,2,\ldots,M+1). \quad (26)$$

From equation (17) we get A_{11} and B_{11} by applying the solvability condition. From the right-hand side of equation (17) the general form of A_{11} and B_{11} can be seen to be:

$$A_{11} = \alpha c_{\mu 1i} b_0; \quad B_{11} = \alpha c_{\mu 1r}. \quad (27)$$

Here $c_{\mu 1r}$ and $c_{\mu 1i}$ are the (mean temperature) first order effects on $\hat{c} = c_r + i c_i$. From equation (18) in a similar way A_{12} and B_{12} emerges as:

$$A_{12} = \alpha c_{\mu 2i} b_0; \quad B_{12} = \alpha c_{\mu 2r}. \quad (28)$$

Here $c_{\mu 2r}$ and $c_{\mu 2i}$ are the temperature fluctuation first order effects on $\hat{c} = c_r + i c_i$. These two parameters will be different for a situation with shape assumption compared to one without it. In order to be precise we added a superscript n to the parameters involved if no shape assumption is made and a superscript s if we make the shape assumption.

Prior to applying the solvability condition, equation (21) is introduced into equation (18), so that the right-hand side of equation (18) now reads:

$$[f_{mn}]_{(M+1)\times(M+1)}[a_{rn} + i a_{in})_{(M+1)\times 1} - (A_{12} - i b_0 B_{12})[g_{rn} + i g_{in}]_{(M+1)\times 1}$$

with A_{12} and B_{12} being of the following form (superscript n: no shape assumption),

$$A^{\mathrm{n}}_{12} = \sum_{n=1}^{M+1} A^1_{12n} a_{rn} + \sum_{n=1}^{M+1} A^2_{12n} a_{in} \quad (29)$$

$$B^{\mathrm{n}}_{12} = \frac{1}{b_0}\left(\sum_{n=1}^{M+1} B^1_{12n} a_{rn} + \sum_{n=1}^{M+1} B^2_{12n} a_{in}\right). \quad (30)$$

Now A^1_{12n}, A^2_{12n}, B^1_{12n} and B^2_{12n} can be determined. According to equation (28) then $c^{\mathrm{n}}_{\mu 2i}$ and $c^{\mathrm{n}}_{\mu 2r}$ are known.

With increasing Prandtl number, we need more Chebyshev polynomials to obtain accurate solutions of equations (12), (22) and (23). For $Pr = 100$, the largest Pr number considered, 44 Chebyshev polynomials are appropriate. In all calculations displayed, 44 Chebyshev polynomials were used.

6. RESULTS AND DISCUSSION

6.1. *Zero-order temperature disturbance*

From the time $t = 0$ at which the velocity disturbance is brought into the flow, the temperature disturbance $\tilde{T}_0 = \hat{T}_0 \exp[i(\alpha x - \theta)] + \mathrm{c.c.}$ develops. For $t \to \infty$, we expect the shape function $\hat{T}_0(y,t)$ to develop towards the form $\hat{T}_0(y)$ according to the shape assumption (1).

In Fig. 2(a–e), the development of $\hat{T}_0(y,t)$ with time is shown for a specific set of parameters ($Pr = 7, Re = 5800, \alpha = 1.02$). The broken line shows $\hat{T}_0$ calculated according to the shape assumption. From these figures, we estimate that 100 time units are necessary before the shape function obtains its final form. Due to the nondimensionalization of $t = t^* U^*_R / H^*$ a typical wave period is about 25 time units. From our calculations, as well as from physical considerations we determined that this adjustment time basically depends on the Peclet number $Pe = RePr$. The Peclet number determines the influence of thermal diffusion and for Pe 10 times smaller than

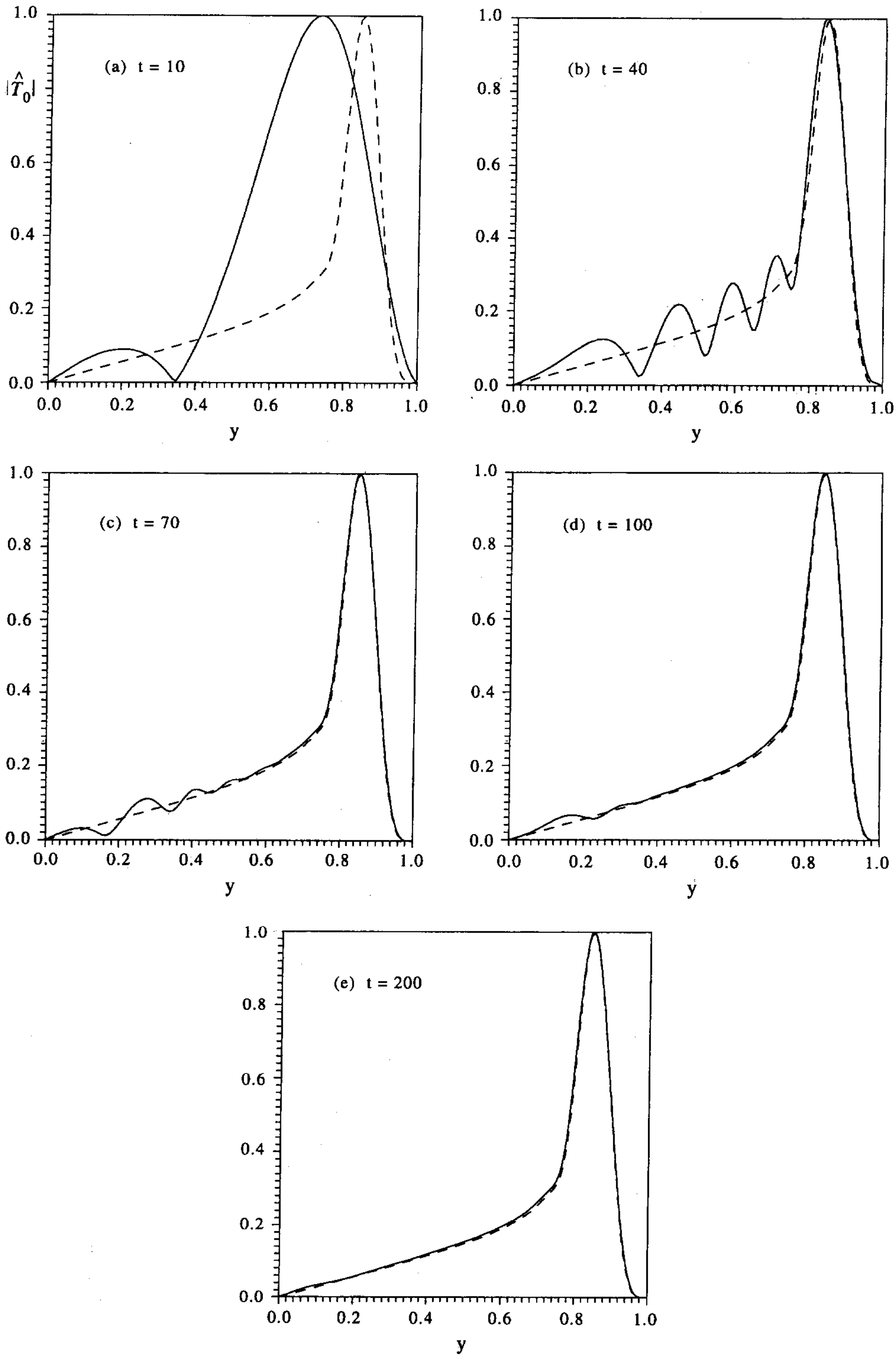

Fig. 2. Shape function $|\hat{T}_0| = \sqrt{T_{0r}^2 + T_{0i}^2}$ at different times, normalized by $\max|\hat{T}_0| = 1.0$, $Pr = 7$, $Re = 5800$, $\alpha = 1.02$. (---) Shape function according to the shape assumption.

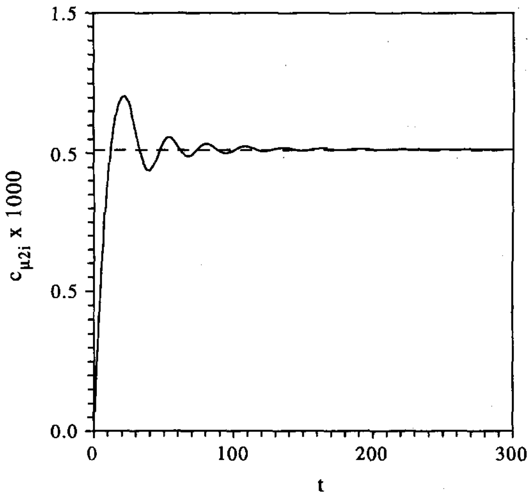

Fig. 3. Amplification parameter $c_{\mu 2i}$ for $Pr = 7$, $\alpha = 1.02$, $Re = 5800$. (—) $c^{n}_{\mu 2i}$, no shape assumption. (- - -) $c^{s}_{\mu 2i}$, shape assumption.

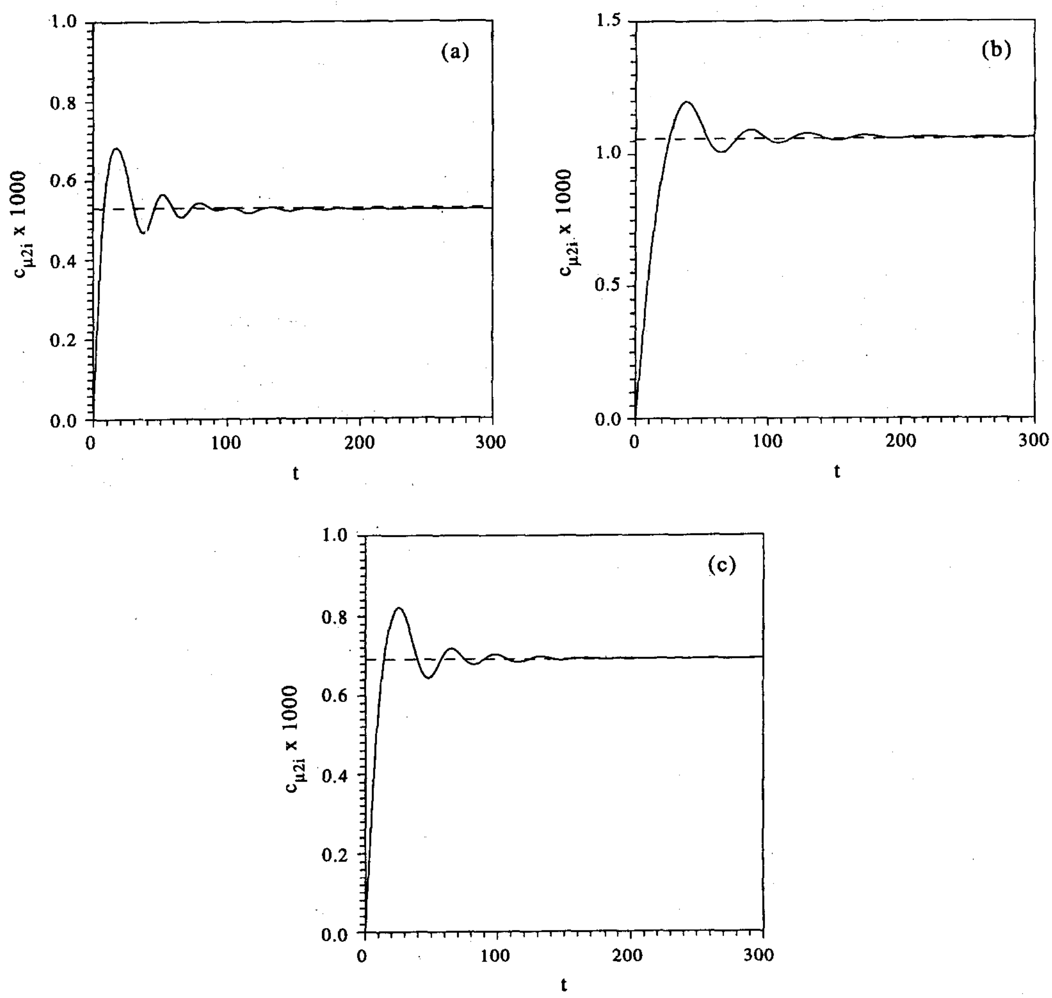

Fig. 4. Amplification parameter $c_{\mu 2i}$ for $Pr = 7$, $Re = 10\,000$. (a) $\alpha = 1.09$; (b) $\alpha = 0.8$; (c) $\alpha = 0.95$. (—) $c^{n}_{\mu 2i}$, no shape assumption. (- - -) $c^{s}_{\mu 2i}$, shape assumption.

that given in Fig. 2, the adjustment time is approximately 20% lower.

Since $\hat{T}_0$ appears on the right-hand side of equation (18), we may expect the shape function $\hat{\psi}_{\mu 2}$ and the parameter $c^n_{\mu 2i}$ to develop as slowly as $\hat{T}_0$ does. However, although $\hat{\psi}_{\mu 2}$ and $c^n_{\mu 2i}$ are both determined from equation (18), only $\hat{\psi}_{\mu 2}$ is found to develop slowly.

6.2. *First-order temperature disturbance effects*

In Fig. 3, the parameter $c^n_{\mu 2i}$ is shown as a function of time for a particular set of parameters. Here, as with all other parameter combinations we investigated, $c^n_{\mu 2i}$ obviously gains its approximate final value very rapidly. Since $c^n_{\mu 2i}$ directly determines how the amplification parameter $c_i = c_{0i} + \varepsilon K_\mu (c_{\mu 1i} + c_{\mu 2i}) + O(\varepsilon^2)$ is affected by temperature fluctuation effects, we see that the shape assumption is correct after very short times. In experiments [9], velocity fluctuations behave according to the shape assumption after about 1.2 wavelengths, which corresponds to about 30 time units in our examples. According to Fig. 3, $c^n_{\mu 2i}$ reaches its final value in approximately the same time. It is possible that this time would be slightly different if we had accounted for the $\hat{\psi}_0$ development as well. However, this would not change the behavior of $c^n_{\mu 2i}$ considerably.

The results shown in Figs. 2 and 3 hold for a parameter combination (α, *Re*) which is close to the critical value. In Fig. 4(a–c) three cases are shown at a higher Reynolds number. Figure 4(a) shows the case with α close to the upper branch of the critical curve, Fig. 4(b) is close to the lower branch and Fig. 4(c) is the case between the upper and lower branches. In all cases $c^n_{\mu 2i}$ has the same rapid approach to its asymptotic values as in Fig. 3.

6.3. *How important are the temperature disturbances?*

Comparing $c_{\mu 2i}$ and $c_{\mu 1i}$ (for large time t, for example) indicates the relative importance of temperature fluctuation effects vs those of the mean temperature. In Fig. 5, both amplification parameters are shown for a large range of Prandtl numbers. The ratio of $c_{\mu 2i}/c_{\mu 1i}$ is well below 0.1 for most Prandtl numbers. This limits the temperature fluctuation effects to less than 10% of the mean temperature influence. For other parameter combinations chosen in the course of our investigation this was also the case. Our final conclusions are then:

(1) It is appropriate to account for temperature fluctuations $\tilde{T}$ by the shape assumption.

(2) If they are neglected completely the error within the temperature effects is of the order of 10%.

Both conclusions are drawn from our study of Poiseuille flow with temperature dependent viscosity. Since the physics underlying temperature fluctuations are fundamentally similar in Poiseuille, channel and boundary layer flows we suggest this study of Poiseuille flow to be an order of magnitude analysis for all flows of this kind. A straightforward method is provided in this study that may be readily applied to other flows

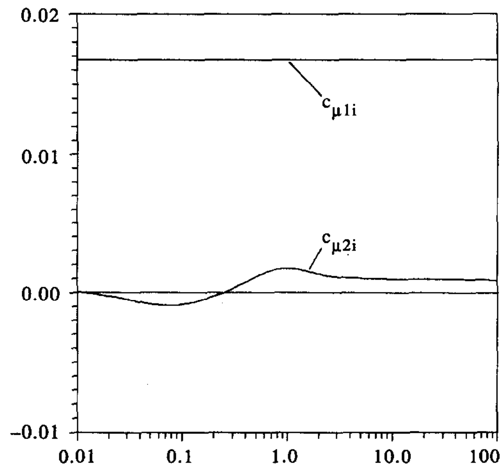

Fig. 5. Amplification parameters $c_{\mu 1i}$ and $c_{\mu 2i}$ for different *Pr* numbers $\alpha = 1.02$, *Re* = 5800.

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